Momentum Balance Along A Flow Streamline

Applying Newton's Second Law to a fixed volume within a steady flow can yield an expression for momentum conservation along a flow streamline. For a steady flow, the momentum contained within a fixed volume must remain constant. The sum of the forces acting on the boundaries of the volume (acting in the direction of the flow) is therefore equal to the rate at which momentum flows out of the volume minus the rate at which the momentum flows in.

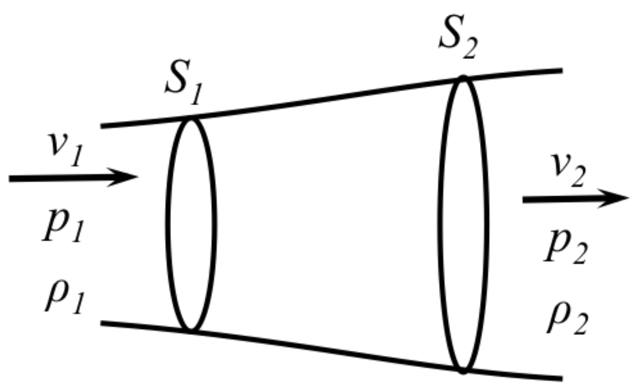

Figure 1. Volume For Determining Momentum Conservation Along A Flow Streamline.

The volume shown in Figure 1 is constructed such that its "sides" are parallel to the flow direction. Because of this construction, fluid enters through the upstream "face" S_1 and exits through the downstream face S_2, but no fluid passes though the sides. If there are no shear forces acting in the fluid (negligible viscosity) then the forces on the boundary are due only to the pressure, and Newton's Second Law can be written as:

(1) $$p_1 S_1 - p_2 S_2 + \sum F_{sides} = (\rho_2 v_2 S_2) v_2 - (\rho_1 v_1 S_1) v_1.$$

If the upstream and downstream faces are placed close enough together so that changes in the velocity and pressure are very small, then: $v_2 = v_1 + \delta v$, $p_2 = p_1 + \delta p$ and $S_2 = S_1 + \delta S$, where $\delta v \ll v_1$, $\delta p \ll p_1$ and $\delta S \ll S_1$.

The forces on the sides of the control volume resolve to:

(2) $$\sum F_{sides} = \frac{p_1 + p_2}{2}(S_2 - S_1) = \left(p_1 + \frac{\delta p}{2}\right)\delta S.$$

With these relations, equation (1) becomes:

(3) $$p_1 S_1 - (p_1 + \delta p)(S_1 + \delta S) + \left(p_1 + \frac{\delta p}{2}\right)\delta S = (\rho_2 v_2 S_2) v_2 - (\rho_1 v_1 S_1) v_1.$$

Conservation of mass requires: $\rho_2 v_2 S_2 = \rho_1 v_1 S_1$. Substituting this on the right hand side yields:

(4) $$p_1 S_1 - (p_1 + \delta p)(S_1 + \delta S) + \left(p_1 + \frac{\delta p}{2}\right)\delta S = (\rho_1 v_1 S_1)(v_1 + \delta v) - (\rho_1 v_1 S_1) v_1.$$

Distributing terms:

(5) $$p_1 S_1 - p_1 S_1 - \delta p S_1 - p_1 \delta S - \delta p \delta S + p_1 \delta S + \frac{\delta p \delta S}{2} = (\rho_1 v_1 S_1)\delta v.$$

Simplifying and keeping only first-order terms:

$$(6) \qquad -\ \delta p S_1 = (\rho_1 v_1 S_1)\delta v.$$

Dropping the subscript and dividing both sides by S:

$$(7) \qquad -\ \delta p = \rho v \delta v.$$

The product rule for differentiation allows us to write equation (7) as:

$$(8) \qquad -\ \delta p = \rho \delta \left(\frac{v^2}{2}\right)$$

If the flow density is constant, then integrating both sides of equation (8) gives:

$$(9) \qquad -\ (p_2 - p_1) = \rho \left(\frac{{v_2}^2}{2} - \frac{{v_1}^2}{2}\right),$$

or:

$$(10) \qquad p_1 + \rho \frac{{v_1}^2}{2} = p_2 + \rho \frac{{v_2}^2}{2}.$$

Which is the familiar form of the Bernoulli Equation.

We can use this equation to find the pressure at the surface of an airfoil by starting from a position of known pressure and velocity and following a flow streamline to the surface of the airfoil. Such a path is shown in figure 2.

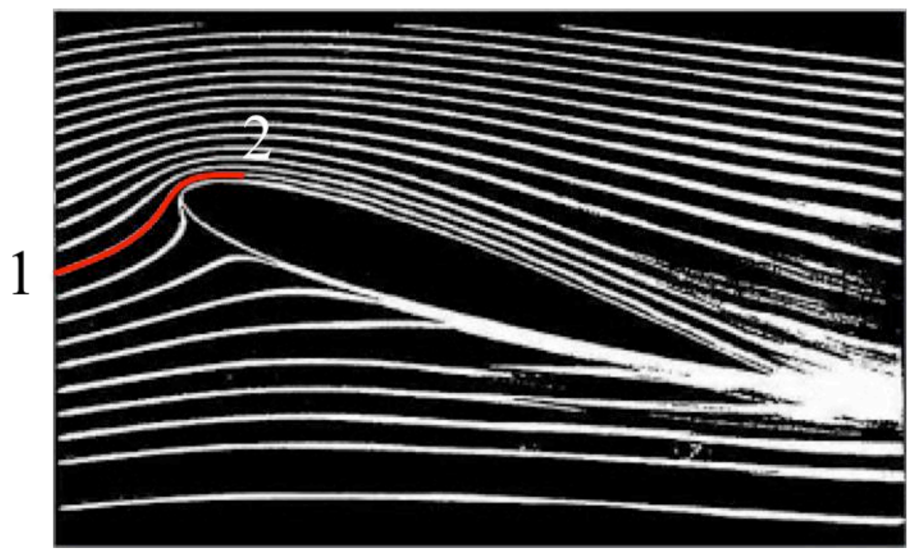

Figure 2. Path To Airfoil Surface Along A Flow Streamline.

If we move point 1 far enough upstream of the airfoil, then the pressure there is the ambient pressure: $p_1 = p_{amb}$ and the velocity is the freestream velocity: $v_1 = V$. The Bernoulli equation becomes:

$$(11) \qquad p_{amb} + \rho \frac{V^2}{2} = p_2 + \rho \frac{{v_2}^2}{2} = \text{constant},$$

or solving for p_2:

(12) $$p_2 = p_{amb} + \rho\frac{V^2}{2} - \rho\frac{{v_2}^2}{2}.$$

By defining the "total pressure" as: $p_T \equiv p_{amb} + \rho\frac{V^2}{2}$ (constant) and the "dynamic pressure": $q \equiv \rho\frac{v^2}{2}$ the pressure at any point on the surface of the airfoil can be written:

(13) $$p = p_T - q.$$

Momentum Balance Perpendicular To Flow Streamlines

Just as we were able to use Newton's Second Law to derive an expression for momentum conservation along a streamline, we can use it to derive an expression for momentum conservation perpendicular to a streamline. We consider the forces acting on a fluid element as shown in figure 3.

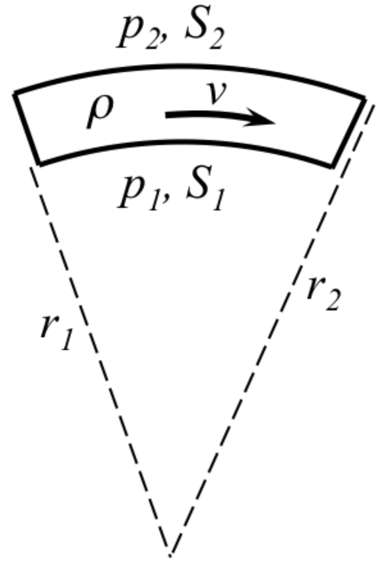

Figure 3. Element For Determining Momentum Conservation Perpendicular To A Flow Streamline.

Newton's Second Law for the fluid element shown in figure 3 is:

(14) $$p_2 S_2 - p_1 S_1 + \sum F_{sides} = ma = \rho V\frac{v^2}{r_1},$$

where *r* is the local radius of curvature of the streamline. For small changes in *r*: $r_2 = r_1 + \delta r$, $p_2 = p_1 + \delta p$ and $S_2 = S_1 + \delta S$, where $\delta r \ll r_1$, $\delta p \ll p_1$ and $\delta S \ll S_1$.

The forces on the sides of the fluid element resolve to:

$$\sum F_{sides} = -\left(p_1 + \frac{\delta p}{2}\right)\delta S.$$

With these relations, Newton's Second Law (equation 14) becomes:

(15) $$(p_1 + \delta p)(S_1 + \delta S) - p_1 S_1 - \left(p_1 + \frac{\delta p}{2}\right)\delta S = \frac{\rho S_1 \delta r v^2}{r_1},$$

or:

$$p_1 S_1 + \delta p S_1 + p_1 \delta S + \delta p \delta S - p_1 S_1 - p_1 \delta S - \frac{\delta p \delta S}{2} = \frac{\rho S_1 \delta r v^2}{r_1}. \tag{16}$$

Collecting terms gives and keeping only first order:

$$\delta p S_1 = \frac{\rho S_1 \delta r v^2}{r_1}. \tag{17}$$

Dividing both sides by *S* and dropping subscripts:

$$\delta p = \frac{\rho v^2 \delta r}{r}, \tag{18}$$

or:

$$\frac{dp}{dr} = \frac{\rho v^2}{r}. \tag{19}$$

We should be able to use this equation to find the pressure at the surface of an airfoil. In this case we would integrate the equation as we moved along a path perpendicular to the flow streamlines as shown in figure 4.

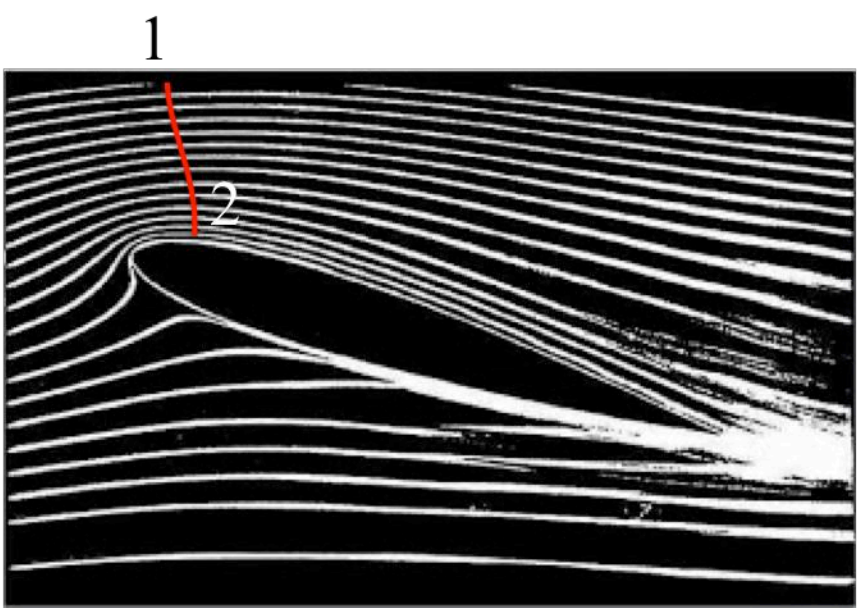

Figure 4. Path To Airfoil Surface Perpendicular To Flow Streamlines.

Equation (19), relating the perpendicular pressure gradient to the local flow curvature does not lend itself to straightforward integration in an arbitrary flow field. Because of this, we first consider simpler flow fields with pure rotation (tangential velocities only).

The first case we consider is a fluid in "solid body rotation". Imagine a steadily spinning bucket of water. When the water reaches equilibrium, the tangential velocity of a water element will be proportional to its distance from the axis of rotation:

(20) $$v_t = kr.$$

The radial pressure gradient in the bucket will be:

(21) $$\frac{dp}{dr} = \frac{\rho {v_t}^2}{r} = \rho k^2 r.$$

We can integrate this expression to determine the pressure:

(22) $$p = \rho k^2 \frac{r^2}{2} + C,$$

where C is a constant. If we sum the static and dynamic pressure at a point in the water in order to get the total pressure, we find:

(23) $$p_T = p + \rho \frac{v^2}{2} = \rho k^2 \frac{r^2}{2} + C + \rho \frac{k^2 r^2}{2} = C + \rho k^2 r^2.$$

Obviously the total pressure varies with r and is not the same throughout the bucket as we might expect from the Bernoulli equation. The Bernoulli equation is not "wrong", it was derived by considering momentum conservation along a streamline. As we move radially through the water, we are moving from one streamline to another, and there is no reason to expect the Bernoulli equation to apply. For the Bernoulli equation to hold from streamline to streamline, it turns out that the flow must be "irrotational". In irrotational flow, if you integrate the velocity around a closed path you get zero:

(24) $$\oint \boldsymbol{v} \cdot \boldsymbol{ds} = 0.$$

We can perform the integration in equation (24) for the case of solid body rotation as shown in figure 5.

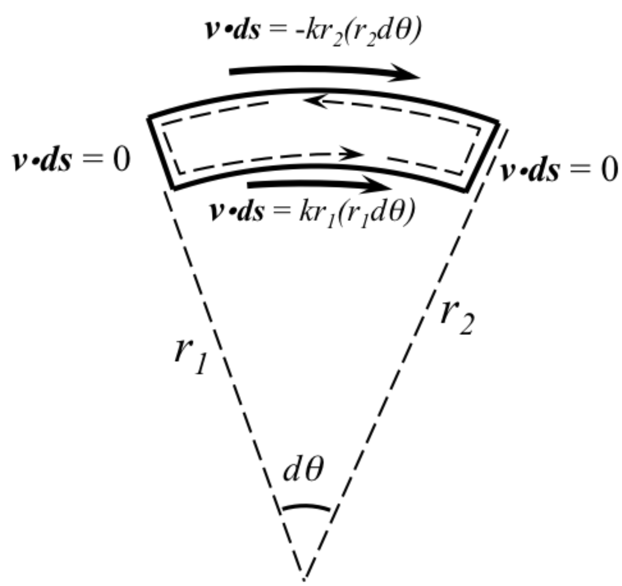

Figure 5. Closed Path Integral For Solid Body Rotation.

An integral around a closed path evaluates to:

$$\oint \boldsymbol{v} \cdot \boldsymbol{ds} = kr_1(r_1 d\theta) + 0 - kr_2(r_2 d\theta) + 0 = kd\theta({r_1}^2 - {r_2}^2) \neq 0. \tag{25}$$

This shows that solid body rotation does not represent irrotational flow.

We now consider a different purely tangential flow where the velocity profile is proportional to $1/r$:

$$v_t = \frac{k}{r}. \tag{26}$$

A closed-path integral for this flow (one that doesn't enclose the point $r = 0$) evaluates to:

$$\oint \boldsymbol{v} \cdot \boldsymbol{ds} = \frac{k}{r_1}(r_1 d\theta) + 0 - \frac{k}{r_2}(r_2 d\theta) + 0 = 0, \tag{27}$$

so a $1/r$ velocity profile is an example of irrotational flow (everywhere except at $r = 0$). The radial pressure gradient in this flow is:

$$\frac{dp}{dr} = \frac{\rho {v_t}^2}{r} = \rho \frac{k^2}{r^3}, \tag{28}$$

which integrates to:

$$p = -\rho k^2 \frac{1}{2r^2} + C. \tag{29}$$

where C is again a constant. If we sum the static and dynamic pressure we get:

$$p_T = p + \rho \frac{v^2}{2} = -\rho k^2 \frac{1}{2r^2} + C + \rho k^2 \frac{1}{2r^2} = C. \tag{30}$$

We find that the total pressure is constant throughout this irrotational flow, consistent with the traditional Bernoulli equation (equation 11).

We can compute the closed-path integral over a small region in order to determine a general condition for irrotational flow. For a very small region, equation (24) evaluates to:

$$\oint \boldsymbol{v} \cdot \boldsymbol{ds} = v_1(r_1 d\theta) + 0 - (v_1 + \delta v)(r_1 + \delta r)d\theta + 0. \tag{31}$$

In order for the flow to be irrotational:

$$v_1 r_1 d\theta - v_1 r_1 d\theta - \delta v r_1 d\theta - v_1 \delta r d\theta - \delta v \delta r d\theta = 0. \tag{32}$$

Dividing by $d\theta$ and keeping only first-order terms we get:

$$-\delta v r - v \delta r = 0, \tag{33}$$

or:

$$\delta r = -\frac{r \delta v}{v}. \tag{34}$$

We recall our equation for perpendicular momentum balance (equation 18):

$$\delta p = \frac{\rho v^2 \delta r}{r} \tag{35}$$

Substituting the condition for irrotational flow (equation 34) gives:

$$\delta p = \frac{\rho v^2 \delta r}{r} = -\frac{\rho v^2}{r}\frac{r\delta v}{v} = -\rho v \delta v \tag{36}$$

This equation is *exactly* the same as the equation we found by considering momentum conservation along a streamline. This shows that in an irrotational flow, you find the same relation between pressure and velocity whether you consider momentum balance along or perpendicular to the flow streamlines. So whether we integrate the pressure gradient along a streamline to get to the surface of an airfoil as in figure 2, or integrate the pressure gradient perpendicular to the flow streamlines as in figure 4, we end up with the same value for the pressure at the surface.

Unsteady Analysis

If we consider the flow around an airfoil (in steady flight) from a frame of reference fixed to the airfoil, the flow is steady. This means that if we pick any point relative to the airfoil and measure the flow conditions there (i.e. pressure, velocity and density), the time-averaged conditions will not change from one moment to the next. On the other hand, if we consider the flow in a reference frame where the air is initially still and the airfoil passes through at constant speed, the flow is unsteady. The pressure and velocity at any point will change as the airfoil passes by. Momentum conservation in an unsteady (but inviscid) flow can be expressed as:

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}\right) = -\nabla p. \tag{37}$$

Where quantities shown in bold represent vectors. The flow in the reference frame where the flow is initially still represents a special case of unsteady flow. By recognizing that the flow in the reference frame of the airfoil is steady, we can rewrite the partial time derivative operator in the unsteady reference frame as:

$$\frac{\partial}{\partial t} = -\boldsymbol{V}\cdot\nabla, \tag{38}$$

where $\boldsymbol{V}$ represents the velocity of the airfoil relative to the still air. With this identity, the expression for momentum conservation (equation 37) becomes:

$$\rho(-\boldsymbol{V}\cdot\nabla\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}) = \rho(-\boldsymbol{V}+\boldsymbol{v})\cdot\nabla\boldsymbol{v} = -\nabla p. \tag{39}$$

We consider the following identity:

$$\nabla(-\boldsymbol{V}+\boldsymbol{v}) = -\nabla\boldsymbol{V} + \nabla\boldsymbol{v}. \tag{40}$$

Because $\boldsymbol{V}$ is a constant, $-\nabla\boldsymbol{V} = 0$, so equation (40) becomes:

$$\nabla(-\boldsymbol{V}+\boldsymbol{v}) = \nabla\boldsymbol{v}. \tag{41}$$

We can use this to write the expression for momentum conservation (equation 39) as:

(42) $$\rho(-\boldsymbol{V}+\boldsymbol{v})\cdot\nabla(-\boldsymbol{V}+\boldsymbol{v}) = -\nabla p,$$

or:

(43) $$\frac{1}{2}\rho\nabla[(-\boldsymbol{V}+\boldsymbol{v})\cdot(-\boldsymbol{V}+\boldsymbol{v})] = -\nabla p.$$

Distributing terms on the left gives:

(44) $$\frac{1}{2}\rho\nabla[V^2 - 2\boldsymbol{V}\cdot\boldsymbol{v} + v^2] = -\nabla p.$$

Noting that $V = \text{constant}$:

(45) $$\frac{1}{2}\rho\nabla[v^2 - 2\boldsymbol{V}\cdot\boldsymbol{v}] = -\nabla p.$$

Integrating both sides of this equation (and assuming uniform density) gives:

(46) $$\rho\left[\frac{1}{2}({v_2}^2 - {v_1}^2) - (\boldsymbol{V}\cdot\boldsymbol{v_2} - \boldsymbol{V}\cdot\boldsymbol{v_1})\right] = -(p_2 - p_1),$$

or:

(47) $$p_1 + \rho\left(\frac{1}{2}{v_1}^2 - \boldsymbol{V}\cdot\boldsymbol{v_1}\right) = p_2 + \rho\left(\frac{1}{2}{v_2}^2 - \boldsymbol{V}\cdot\boldsymbol{v_2}\right).$$

We can compare equation (47) to the Bernoulli equation (equation 11) that is valid in steady flow. If $\boldsymbol{v}'$ is the velocity in the reference frame fixed to the wing then:

(48) $$\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{V}.$$

This vector relationship is shown graphically in figure 5.

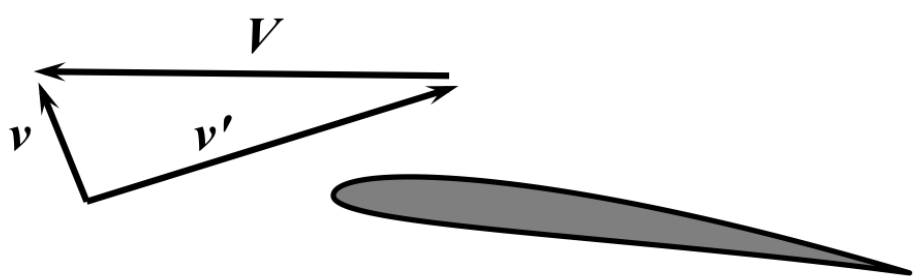

Figure 5. Relationship Between Airfoil-Relative And Still Air-Relative Velocities.

With the substitution of equation (48), the left hand side of equation (47) becomes:

(49) $$p_1 + \rho\left[\frac{1}{2}(\boldsymbol{v'_1}+\boldsymbol{V})\cdot(\boldsymbol{v'_1}+\boldsymbol{V}) - \boldsymbol{V}\cdot(\boldsymbol{v'_1}+\boldsymbol{V})\right].$$

Distributing terms:

(50) $$p_1 + \rho\left[\tfrac{1}{2}\left({v'_1}^2 + 2\boldsymbol{V}\cdot\boldsymbol{v'_1} + V^2\right) - \boldsymbol{V}\cdot\boldsymbol{v'_1} - V^2\right].$$

Collecting terms and simplifying, the left hand side of equation (47) simplifies to:

(51) $$p_1 + \frac{1}{2}\rho\left[{v'_1}^2 - V^2\right].$$

The full expression relating pressure and velocity (equation 47) then becomes:

$$(52) \qquad p_1 + \frac{1}{2}\rho\left[{v'_1}^2 - V^2\right] = p_2 + \frac{1}{2}\rho\left[{v'_2}^2 - V^2\right],$$

or:

$$(53) \qquad p_1 + \frac{1}{2}\rho {v'_1}^2 = p_2 + \frac{1}{2}\rho {v'_2}^2,$$

which is simply the steady form of the Bernoulli equation expressed in terms of the velocities in the (steady) airfoil-fixed reference frame.

If we choose *x* direction the same as the airfoil's direction of motion then equation (47) becomes:

$$(54) \qquad p_1 + \rho\left(\frac{1}{2}{v_1}^2 - V{v_1}_x\right) = p_2 + \rho\left(\frac{1}{2}{v_2}^2 - V{v_2}_x\right).$$

If we take p_2 to be the pressure where the air is still then:

$$(55) \qquad p_2 = p_{amb}, \;\; v_2 = 0,$$

and we find:

$$(56) \qquad p_1 + \rho\left(\frac{1}{2}{v_1}^2 - V{v_1}_x\right) = p_{amb},$$

or at any point in the flow:

$$(57) \qquad p = p_{amb} + \rho\left(Vv_x - \frac{1}{2}v^2\right).$$

Equation (57) can be used to demonstrate two important concepts:

1. As a lifting wing passes overhead, the pressure on the ground increases even though the air there has been accelerated from rest (v_x is positive, but less than *V*). The "Bernoulli Truism" (increasing velocity implies decreasing pressure) does not hold in unsteady flow.
2. The pressure at the surface of an airfoil can be computed using either the velocity measured relative to the airfoil or relative to the still air. However, to compute the pressure using flow velocity relative to the airfoil you must use equation (11). To compute the pressure using the flow velocity relative to the still air, you must use equation (57). For example, the magnitude of the inviscid flow velocity at the top of a (non-rotating) circular cylinder is $2V$ measured relative to the cylinder. Equation (11) therefore indicates the pressure at the top of the cylinder is: $p_{amb} + \frac{1}{2}\rho V^2 - \frac{1}{2}\rho(2V)^2 = p_{amb} - \frac{3}{2}\rho V^2$. The magnitude of the flow velocity at the top of the cylinder is –*V* measured relative to the still air. Equation (57) indicates the pressure at the top of the cylinder is: $p_{amb} + \rho\left(-V^2 - \frac{1}{2}V^2\right) = p_{amb} - \frac{3}{2}\rho V^2$. The same analysis can be done at the front of the cylinder (the forward stagnation point). The magnitude of the flow velocity at the front of the cylinder is 0 relative to the cylinder. Equation (11) therefore indicates the pressure there is: $p_{amb} + \frac{1}{2}\rho V^2$. The magnitude of the flow velocity at the front of the cylinder is *V* relative to the still air. Equation (57) indicates the pressure there is: $p_{amb} + \rho\left(V^2 - \frac{1}{2}V^2\right) = p_{amb} + \frac{1}{2}\rho V^2$. You can compute

the pressure on a cylinder or airfoil in either reference frame, but you must use the correct equation to relate changes in velocity to changes in pressure.